

n-particle sector of field theory as a quantum open system.

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Abstract

We give an exposition of a technique, based on the Zwanzig projection formalism, to construct the evolution equation for the reduced density matrix corresponding to the n-particle sector of a field theory. We consider the case of a scalar field with a $g\phi^3$ interaction as an example and construct the master equation at the lowest non-zero order in perturbation theory.

1 Motivation

The Hilbert space of a quantum field theory is the Fock space \mathcal{F} constructed from the n-particle subspaces H^n as $\mathcal{F} = \bigoplus_{n=0}^{\infty} H^n$. It is a well known feature that the interaction terms cause mixing of the states between these subspaces.

It is the aim of this letter to offer a treatment of the subspaces H^n as corresponding to an open quantum system, that is we want to obtain an evolution equation for the reduced density matrix corresponding to the n-particle subspace, while treating the rest of the Fock space as an “environment”. Our master equation is expected to contain (in general non-local in time) terms describing dissipation and noise. In addition, the fact that at any time the

system might leave the relevant subspace the density matrix will not be normalized to one. The quantity $1 - \text{Tr} \rho_{rel}(t)$ will correspond to the probability that at time t the n -particle subspace is “empty”.

This is a problem of interest mainly for two reasons. First, in an attempt to explain the manifestation of particles as spatially localized excitations it is important to know the degree of validity of the particle description. Our approach of treating the n -particle sector as an open system provides measures of the modification in the equations of motion (dissipation) as well as of unpredictability (noise) inherent in such a coarse grained description.

Our second motivation is the necessity of having a technique reducing the description of a field system at the particle level. This is important in the context of studying open field-theoretic quantum systems. We can derive master equations describing the evolution of reduced density matrix of a field system, by tracing out over the states of the environment. The environment can for instance be a thermal bath or another field. We are then naturally led to the question of what the evolution according to the master equation will imply for the particles corresponding to the field. Our preoccupation is particularly with the case of QED, where we can obtain the evolution equation for the spinor field by tracing out the states of the photon field [3]. The translation of this into a particle description is related to the notion of environment induced decoherence [1] and the possibility of dynamical origin of the charge superselection rules [2].

In this letter, we are primarily interested in the mathematical aspects of the reduction of the field description into the particle level. We are therefore going to consider the simple case of a $g\phi^3$ scalar field theory, even though we do not expect to obtain results of any physical interest. We should stress though, that the technique developed here is easily generalized to apply for any field theory that can be treated perturbatively and for all (even non-unitary) dynamics.

2 Preliminaries

The Zwanzig technique For the construction of the master equation we employ the Zwanzig projection technique (for review and discussion see [4, 6, 7]). The main concept in this formalism is the representation of the coarse graining operation by an indempotent mapping \mathbf{P} in the space of

states

$$\rho \rightarrow \rho_{rel} = \mathbf{P}\rho \quad (2.1)$$

The irrelevant part of the state is then given by

$$\rho_{irr} = (\mathbf{1} - \mathbf{P})\rho \quad (2.2)$$

Considering unitary dynamics for the full theory

$$i\frac{\partial\rho}{\partial t} = \mathbf{L}\rho \equiv [H, \rho] \quad (2.3)$$

we can show [4] that the evolution equation for the relevant part is given by

$$i\frac{\partial\rho_{rel}(t)}{\partial t} = \mathbf{P}\mathbf{L}\rho_{rel}(t) + \mathbf{P}\mathbf{L}e^{-i(1-\mathbf{P})\mathbf{L}t}\rho_{irr}(0) - i\int_0^t d\tau \mathbf{G}(\tau)\rho_{rel}(t-\tau) \quad (2.4)$$

Here \mathbf{G} stands for the kernel

$$\mathbf{G}(\tau) = \mathbf{P}\mathbf{L}(\mathbf{1} - \mathbf{P})e^{-i\mathbf{L}\tau}(\mathbf{1} - \mathbf{P})\mathbf{L}\mathbf{P} \quad (2.5)$$

In equation (3.) the third term corresponds to non-local in time noise and dissipation while the second is essentially a time dependent “external force” acting on the relevant part of the state.

In order to apply the Zwanzig method to our problem we need consider coarse-graining operations of the form

$$\rho_{rel} = \mathbf{P}\rho = P_n\rho P_n \quad (2.6)$$

where P_n is the projection operator corresponding to the n-particle subspace. The form of these operators is particularly transparent in the Bargmann representation, in which we are going to carry our calculations. We here make a small digression for purposes of establishing notation and assembling a number of useful formulas [5].

The Bargmann representation In the Bargmann representation one can associate to each operator \hat{A} on \mathcal{F} a functional of $\alpha(\mathbf{x})$ $\alpha^*(\mathbf{x})$ where $\alpha(\mathbf{x}), \alpha^*(\mathbf{x})$ are elements of H (complex valued functions on a Cauchy hypersurface *Sigma* of Minkowski spacetime). We actually have two choices for these functionals

$$\tilde{A}(\alpha^*, \alpha) = \sum \frac{1}{(n!m!)^{1/2}} \int A_{mn}(\mathbf{x}_1, \dots, \mathbf{x}_m \mid \mathbf{y}_1, \dots, \mathbf{y}_n) \alpha^*(\mathbf{x}_1) \dots \alpha^*(\mathbf{x}_m) \alpha(\mathbf{y}_n) \dots \alpha(\mathbf{y}_1) d^m x d^n y \quad (2.7)$$

in terms of the distributions A_{mn} of the matrix representation of the operator, and

$$A(\alpha^*, \alpha) = \sum \int A_{mn}(\mathbf{x}_1, \dots, \mathbf{x}_m | \mathbf{y}_1, \dots, \mathbf{y}_n) \alpha^*(\mathbf{x}_1) \dots \alpha^*(\mathbf{x}_m) \alpha(\mathbf{y}_1) \dots \alpha(\mathbf{y}_n) d^m x d^n y \quad (2.8)$$

in terms of the distributions K_{mn} appearing in the normal representation of the operator \hat{A}

$$\hat{A} = \sum \int K_{mn}(\mathbf{x}_1, \dots, \mathbf{x}_n | \mathbf{y}_1, \dots, \mathbf{y}_m) \hat{a}^\dagger(\mathbf{x}_1) \dots \hat{a}^\dagger(\mathbf{x}_n) \hat{a}(\mathbf{y}_1) \dots \hat{a}(\mathbf{y}_m) d^m x d^n y \quad (2.9)$$

where $a(\mathbf{x})$ and $a^\dagger(\mathbf{x})$ satisfy the commutation relations

$$[\hat{a}(\mathbf{x}), \hat{a}^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}') \quad (2.10)$$

The two functionals are related by

$$A(\alpha^*, \alpha) = \tilde{A}(\alpha^*, \alpha) \exp \left[- \int \alpha^*(\mathbf{x}) \alpha(\mathbf{x}) dx \right] \quad (2.11)$$

The functional corresponding to the product of operators $\hat{C} = \hat{A}\hat{B}$ is given by

$$\tilde{C}(\alpha^*, \alpha) = \int D\beta^* D\beta \tilde{A}(\alpha^*, \beta) \tilde{B}(\beta^*, \alpha) e^{-\beta^* \beta} \quad (2.12)$$

In terms of the creation and annihilation operators $\hat{a}(\mathbf{x})$ and $\hat{a}^\dagger(\mathbf{x})$ the Hamiltonian for a free field reads

$$\hat{H}_0 = \frac{1}{2} \int dx dx' \hat{a}^\dagger(\mathbf{x}) h(\mathbf{x}, \mathbf{x}') \hat{a}(\mathbf{x}') \quad (2.13)$$

with

$$h(\mathbf{x}, \mathbf{x}') = \int dk e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \omega_{\mathbf{k}} \quad (2.14)$$

Here dk stands for the measure $d^3k/(2\pi)^3$.

The evolution operator $\hat{U}_0(t) = e^{-it\hat{H}_0}$ reads in functional form

$$\tilde{U}_0(\alpha^*, \alpha; t) = \exp \left[\int dx \alpha^*(\mathbf{x}) \Delta(\mathbf{x} - \mathbf{x}'; t) \alpha(\mathbf{x}') \right] \quad (2.15)$$

where

$$\Delta(\mathbf{x} - \mathbf{x}'; t) = \int d\mathbf{k} e^{-i\mathbf{k}(\mathbf{x} - \mathbf{x}')} e^{-i\omega_{\mathbf{k}} t} \quad (2.16)$$

We also note the fundamental formula of Gaussian functional integration

$$\begin{aligned} \int D\beta^* D\beta \beta^*(\mathbf{x}_1) \dots \beta^*(\mathbf{x}_m) \beta(\mathbf{x}'_1) \dots \beta(\mathbf{x}'_n) e^{-\beta^* \beta + f^* \beta + \beta^* f} \\ = \frac{\delta^{m+n}}{\delta f(\mathbf{x}_1) \dots \delta f(\mathbf{x}_m) \delta f(\mathbf{x}'_1) \dots \delta f(\mathbf{x}'_n)} e^{f^* f} \end{aligned} \quad (2.17)$$

Index notation It is more convenient to represent the functional using an index notation. To any function or distribution assign an abstract index to each of its arguments. The index is lower or upper according to whether the corresponding argument is integrated out with an α or an α^* respectively. Hence we represent $\alpha(\mathbf{x})$ with α^a and $\alpha^*(\mathbf{x})$ with α_b^* and $\Delta(\mathbf{x} - \mathbf{x}'; t)$ with Δ_a^b . For example, the potential operator to encounter in the following

$$\hat{V} =: \int dx \frac{g}{3!} \hat{\phi}^3 : \quad (2.18)$$

can be represented as

$$V(\alpha^*, \alpha) = \frac{g}{3!} \left(V_{abc} \alpha^a \alpha^b \alpha^c + 3\alpha_a^* V_{bc}^a \alpha^b \alpha^c + 3\alpha_a^* \alpha_b^* V_c^{ab} \alpha^c + \alpha_a^* \alpha_b^* \alpha_c^* V^{abc} \right) \quad (2.19)$$

with the correspondence

$$V_{abc} \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{-i(\mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (2.20)$$

$$V_{bc}^a \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{-i(-\mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (2.21)$$

$$V_c^{ab} \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{-i(-\mathbf{k}_1 \mathbf{x}_1 - \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (2.22)$$

$$V_{abc} \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{i(\mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (2.23)$$

The inversion of the indices essentially amounts to the complex conjugation of the distribution.

3 The master equation

Now in the Bargmann representation the projection operator \hat{P}_n corresponds to the functional

$$\tilde{P}_n(\alpha^*, \alpha) = \frac{1}{n!}(\alpha^* \alpha)^n \quad (3.1)$$

Substituting this into the pre-master equation(3.8) we obtain

$$\begin{aligned} i \frac{\partial \hat{\rho}(t)}{\partial t} &= [\hat{H}_0, \hat{\rho}(t)] + [\hat{P}_n \hat{V} \hat{P}_n, \hat{\rho}(t)] \\ &\quad - i \int_0^t d\tau \left[-\hat{A}_n(\tau) \hat{\rho}(t-\tau) \hat{B}_n^\dagger(\tau) - \hat{B}_n(\tau) \hat{\rho}(t-\tau) \hat{A}_n^\dagger(\tau) \right. \\ &\quad \left. + \hat{C}_n(\tau) \hat{\rho}(t-\tau) \hat{C}_n^\dagger(-\tau) + \hat{C}_n^\dagger(\tau) \hat{\rho}(t-\tau) \hat{C}_n^\dagger(-\tau) \right] + \hat{F}_{res}(t) \end{aligned} \quad (3.2)$$

where the operator-valued kernels \hat{A} , \hat{B} , \hat{C} read

$$\hat{A}_n(t) = \hat{P}_n \hat{V} (1 - \hat{P}_n) e^{-i(\hat{H}_0 + \hat{V})t} (1 - \hat{P}_n) \hat{V} \hat{P}_n \quad (3.3)$$

$$\hat{B}_n(t) = \hat{P}_n e^{-i(\hat{H}_0 + \hat{V})t} \hat{P}_n \quad (3.4)$$

$$\hat{C}_n(t) = \hat{P}_n \hat{V} (1 - \hat{P}_n) e^{-i(\hat{H}_0 + \hat{V})t} \hat{P}_n \quad (3.5)$$

and the “residual force” operator $\hat{F}_{res}(t)$ corresponding to the effect of the initial irrelevant part of the state is

$$\begin{aligned} \hat{F}_{res}(t) &= \hat{P} \hat{V} (1 - \hat{P}) e^{-i(\hat{H}_0 + \hat{V})t} \hat{\rho}_{irr}(0) e^{i(\hat{H}_0 + \hat{V})t} \hat{P} \\ &\quad - \hat{P} e^{-i(\hat{H}_0 + \hat{V})t} \hat{\rho}_{irr}(0) e^{i(\hat{H}_0 + \hat{V})t} (1 - \hat{P}) \hat{V} \hat{P} \end{aligned} \quad (3.6)$$

For our choice of initial condition, that the system lies within the n-particle subspace at $t = 0$, the residual force term vanishes and our system of equations becomes autonomous.

The lowest order in perturbation expansion The lowest order in the perturbation expansion is obtained by replacing the operator e^{-iHt} in our expressions for the kernel for its free counterpart $e^{-iH_0 t}$.

Let us see what happens to the non-local terms when making this approximation. The operator \hat{C}_n vanishes since P commutes with $e^{-iH_0 \tau}$ and

hence can be made to act on $1 - P$. Hence the non-local term in the master equation can be written

$$-i \int_0^t d\tau \left(-P_n V(1 - P_n)(e^{-iH_0\tau} V e^{iH_0\tau}) P_n \rho(t - \tau) e^{iH_0\tau} P_n \right. \\ \left. - P_n e^{-iH_0\tau} \rho(t - \tau) e^{iH_0\tau} e^{-iH_0\tau} V e^{iH\tau} (1 - P) V P_n \right) \quad (3.7)$$

Now $\|e^{-iH_0\tau} \rho(t - \tau) e^{iH_0\tau} - \rho(t)\| = O(g)$, hence by restricting ourselves to the second order in the coupling constant we get a master equation that is local in time

$$i \frac{\partial \rho}{\partial t} = [H_0, \rho] + [P_n V P_n, \rho] + i \left(L_n(t) \rho + \rho L_n^\dagger(t) \right) + F_{res}(t) \quad (3.8)$$

Here $L_n(t)$ stands for the operator:

$$\hat{L}_n(t) = P_n V(1 - P_n) \left[\int_0^t d\tau V(-\tau) \right] P_n \quad (3.9)$$

where $V(\tau)$ stands for the time evolution of the operator V according to the free Hamiltonian.

We now set forth to calculate \hat{L} for the particular cases of $n = 1$ and $n = 2$. For this we need an expression for the operator $\hat{W}(t) = \int_0^t d\tau V(-\tau)$. The corresponding functional reads

$$\tilde{W}(t)(\alpha^*, \alpha) = \frac{g}{3!} \left(W_{abc} \alpha^a \alpha^b \alpha^c + 3\alpha_a^* W_{bc}^a \alpha^b \alpha^c + 3\alpha_a^* \alpha_b^* W^{ab}{}_c \alpha^c + \alpha_a^* \alpha_b^* \alpha_c^* W^{abc} \right) \quad (3.10)$$

with

$$W_{abc} \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{k_i})^{1/2}} e^{-i(\mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} \\ \times \frac{e^{-i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{-i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (3.11)$$

$$W_{bc}^a \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{-i(-\mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} \\ \times \frac{e^{-i(-\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{-i(-\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (3.12)$$

$$W^{ab}{}_c \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{-i(-\mathbf{k}_1 \mathbf{x}_1 - \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} \\ \times \frac{e^{-i(-\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{-i(-\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (3.13)$$

$$W^{abc} \rightsquigarrow \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} e^{i(\mathbf{k}_1 \mathbf{x}_1 + \mathbf{k}_2 \mathbf{x}_2 + \mathbf{k}_3 \mathbf{x}_3)} \\ \times \frac{e^{i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (3.14)$$

We can easily calculate

$$\begin{aligned} \widetilde{P_1 V}(\alpha^*, \alpha) &= \frac{g}{3!} \left((\alpha^* \alpha) V_{abc} \alpha^a \alpha^b \alpha^c + 3\alpha_a^* V_{bc}^a \alpha^b \alpha^c \right) \\ \widetilde{P_2 V}(\alpha^*, \alpha) &= \frac{g}{3!} \left(\frac{1}{2} (\alpha^* \alpha)^2 + V_{abc} \alpha^a \alpha^b \alpha^c \right. \\ &\quad \left. + 3(\alpha^* \alpha) \alpha_a^* V_{bc}^a \alpha^b \alpha^c + 3\alpha_a^* \alpha_b^* V^{ab}{}_c \alpha^c \right) \end{aligned} \quad (3.15)$$

and from that with repeated application of equation (2.17) we get

$$\tilde{L}_1(t)(\alpha^*, \alpha) = \frac{g^2}{2} \left(\frac{1}{3!} V_{abc} W^{abc} + \alpha_a^* (V_{cd}^a W_{bc}^{cd} + V_{bcd} W^{cda}) \alpha^b \right) \quad (3.16)$$

and

$$\begin{aligned} \tilde{L}_2(t)(\alpha^*, \alpha) &= \frac{g^2}{4} \left(\frac{1}{3} (\alpha^* \alpha)^2 V_{abc} W^{abc} \right. \\ &\quad \left. + (\alpha^* \alpha) \alpha_a^* (2V_{cd}^a W_{bc}^{cd} + V_{bcd} W^{cda}) \alpha^b \right. \\ &\quad \left. + 2 \alpha_a^* \alpha_b^* (2V_{ce}^b W_{ad}^{ea} + V_{cde} W^{abe}) \alpha^c \alpha^d \right) \end{aligned} \quad (3.17)$$

Renormalization Since we performed a perturbation expansion, it is unavoidable to encounter divergences. We readily see that the term $Z = V_{abc} W^{abc}$ as a contraction of two distribution has a δ -function divergence

$$Z(t) = \int \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})} \frac{e^{-i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{-i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} (2\pi)^6 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(0) \quad (3.18)$$

These terms can be removed from the expression of the operator $\hat{L}(t)$ by a renormalization of the density matrix. To see this it is sufficient to remark

that the divergent part of \hat{L} is proportional to the projection operator P_n . Hence in the master equation these divergencies are contained in the terms

$$i\frac{g^2}{24}(Z(t) + Z^*(t))\rho(t) \quad (3.19)$$

These terms can be absorbed through a time dependent renormalization of the density matrix (essentially wave function renormalization [5])

$$\rho(t) \rightarrow \rho_{ren}(t) = \exp\left(\frac{g^2}{24} \int_0^t d\tau [Z(\tau) + Z^*(\tau)]\right) \rho(t) \quad (3.20)$$

which clearly presents positivity, since the term in the exponent is real.

This is not the only term giving rise to infinities in the master equation, but it is more convenient to deal with the others when bringing the master equation into a suitable form.

Having removed these terms we get the following expressions for the operators

$$\begin{aligned} \tilde{L}_1(t) &= \frac{g^2}{2} \alpha_a^* F_b^a \alpha^b \\ \tilde{L}_2(t) &= \frac{g^2}{4} (\alpha^* \alpha) \alpha_a^* F_b^a \alpha^b + \alpha_a^* \alpha_b^* G_{cd}^{ab} \alpha^c \alpha^d \end{aligned} \quad (3.21)$$

with

$$F_b^a \rightsquigarrow \prod_{i=1}^3 \frac{dk_i}{(2\omega_{\mathbf{k}_i})} e^{-i\mathbf{k}_1(\mathbf{x}-\mathbf{x}')} (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \quad (3.22)$$

$$\times \left(\frac{e^{-i(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})t} - 1}{-i(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})} + \frac{e^{i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{i(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} \right) \quad (3.23)$$

$$\begin{aligned} G_{cd}^{ab} &\rightsquigarrow \int \prod_{i=1}^2 \frac{dk_i}{(2\omega_{\mathbf{k}_i})^{1/2}} \frac{dk'_i}{(2\omega_{\mathbf{k}'_i})^{1/2}} \frac{dk_3}{2\omega_{\mathbf{k}_3}} \\ &\times (2\pi)^6 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(\mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1 - \mathbf{k}_2) e^{i(\mathbf{k}_1\mathbf{x} + \mathbf{k}_2\mathbf{x}' - \mathbf{k}'_1\mathbf{y} - \mathbf{k}'_2\mathbf{y}')} \\ &\times \left(\frac{e^{-i(\omega_{\mathbf{k}'_1} + \omega_{\mathbf{k}'_2} + \omega_{\mathbf{k}_3})t} - 1}{-i(\omega_{\mathbf{k}'_1} + \omega_{\mathbf{k}'_2} + \omega_{\mathbf{k}_3})} + 2 \frac{e^{-i(-\omega'_{\mathbf{k}_1} - \omega'_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})t} - 1}{-i(-\omega'_{\mathbf{k}_1} - \omega'_{\mathbf{k}_2} + \omega_{\mathbf{k}_3})} \right) \end{aligned} \quad (3.24)$$

4 One particle

Let us now concentrate to the case of the one particle, and in particular try to establish the form to which the master equation (3.8) reduces in the non-relativistic regime.

When restricting to the one-particle subspace the reduced density matrix is of the form

$$\tilde{\rho} = \alpha_a^* \rho^a_b \alpha^a \quad (4.1)$$

with ρ^a_b essentially corresponding the standard form of the density matrix $\rho(\mathbf{x}, \mathbf{x}')$ in the position representation. Substituting (4.1), (3.) and (3.) in the master equation (3.18) we obtain the evolution equation

$$i \frac{\partial}{\partial t} \rho^a_b = h^a_c \rho^c_b - \rho^c_b h^a_c + i \frac{g^2}{2} (F^a_c \rho^c_b + \rho^c_b F^a_c) \quad (4.2)$$

It is more convenient to consider the density matrix in the momentum representation

$$\rho(\mathbf{k}, \mathbf{k}') = \int dx dx' e^{-i\mathbf{k}\mathbf{x} + i\mathbf{k}'\mathbf{x}'} \rho(\mathbf{x}, \mathbf{x}') \quad (4.3)$$

in which we obtain the transparent form

$$i \frac{\partial}{\partial t} \rho(\mathbf{k}, \mathbf{k}') = (\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}) \rho(\mathbf{k}, \mathbf{k}') + i g^2 [F(\mathbf{k}, t) + F^*(\mathbf{k}', t)] \rho(\mathbf{k}, \mathbf{k}') \quad (4.4)$$

where the function F is given by

$$F(\mathbf{k}, t) = \frac{i}{8\omega_{\mathbf{k}}} \int \frac{dk'}{\omega_{\mathbf{k}'} \omega_{\mathbf{k}+\mathbf{k}'}} \times \frac{e^{-i\omega_{\mathbf{k}}t} (\omega_{\mathbf{k}} \cos(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}+\mathbf{k}'}t) + i(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}+\mathbf{k}'}t) \sin(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}+\mathbf{k}'}t) - \omega_{\mathbf{k}})}{\omega_{\mathbf{k}}^2 - (\omega_{\mathbf{k}'} + \omega_{\mathbf{k}+\mathbf{k}'})^2} \quad (4.5)$$

The function F is actually divergent, but the divergence (logarithmic) is essentially contained in its $\mathbf{k} = 0$ argument. Therefore, we can define the renormalized function $F_{ren}(\mathbf{k}, t) = F(\mathbf{k}, t) - F(0, t)$ and absorb the divergent part ($F(0, t) + F^*(0, t) = Z'(t)$) in a renormalization of the density matrix

$$\rho(t) \rightarrow \rho_{ren}(t) = \exp \left(g^2 \int_0^t Z'(\tau) d\tau \right) \rho(t) \quad (4.6)$$

The non-relativistic limit To obtain the non-relativistic limit it is sufficient to obtain the lowest order term in the expansion of $F_{ren}(\mathbf{k}, t)$ in powers of $b f \mathbf{k}$ that is, take the approximation

$$F_{ren}(\mathbf{k}, t) \approx \frac{\partial^2 F}{\partial k^i \partial k^j} \Big|_{\mathbf{k}=0} k^i k^j \quad (4.7)$$

since the first order derivative of $F(\mathbf{k}, t)$ can easily be shown to vanish due to symmetry arguments. Writing

$$\frac{\partial^2 F}{\partial k^i \partial k^j} \Big|_{\mathbf{k}=0}(t) = \gamma(t) \delta^{ij} + i \zeta(t) \delta^{ij} \quad (4.8)$$

as we expect again from symmetry requirements, we obtain the master equation

$$i \frac{\partial}{\partial t} \hat{\rho} = (1 + 2im g^2 \zeta(t)) [\hat{H}_0, \hat{\rho}] + i 2m g^2 \{ \hat{H}_0, \hat{\rho} \} \quad (4.9)$$

Some comments The form of equation (4.9) is clearly unusual, but the interpretation is not difficult. Essentially, the effect of the environment is to complexify the Hamiltonian, a fact which relates to the non-conservation of particles (it is easy to check that this is the term that fails to preserve the trace). The term in the anticommutator is a term exhibiting dissipation through coupling to energy.

Also the fact that we have considered a three-field interaction vertex leads to the non-appearance of a potential term in the master equation. This will be expected to appear when considering at least the $n = 3$ level. To obtain the non-relativistic potential one has to consider theories with two relevant field lines at the vertex, as is the case of QED.

References

- [1] P. M. V. Barone and A. O. Caldeira, *Phys. Rev. A* **43**, 57 (1991).
- [2] D. Giulini, C. Kiefer and H. D. Zeh, Freiburg preprint THEP-94/30.
- [3] C. Anastopoulos and A. Zoupas, *Master equation for the reduced dynamics of spinor field in QED*, (in preparation).

- [4] H. D. Zeh, *The Physical Basis of the Direction of Time*, (Springer Verlag, Berlin, 1989).
- [5] F. A. Berezin, *The Method of Second Quantization*, (Academic Press, New York, 1966).
- [6] K. Lindenberg and B. J. West, *The non-equilibrium statistical mechanics of open and closed systems*, (VHC Publishers, New Yorkn 1990)
- [7] J. Rau and B. Muller, *From reversible quantum microdynamics to irreversible quantum transport*, nucl-th 9505009.